Problem 1. A fixed point of function $f$ is a number $x^*$ satisfying

$$f(x^*) = x^*.$$ 

The functional iteration is defined as: $x_{n+1} = f(x_n)$, $n = 0, 1, 2, \ldots$ for a given initial approximation $x_0$. Suppose that $f$ is arbitrarily differentiable on an interval $[a, b]$.

i. State and justify sufficient condition(s) under which there exists a fixed point in $[a, b]$.

ii. State and justify additional condition(s) under which there exists a unique fixed point in $[a, b]$.

iii. Prove that, under the conditions stated in Question ii (or state new conditions if necessary), for any $x_0 \in [a, b]$ the functional iteration produces a sequence of points $\{x_n\}$ which converges to the fixed point of $f$, and the convergence rate is at least linear.

Problem 2.

(a) Find the coefficients and nodes of a Gaussian quadrature formula of the form

$$\int_{a}^{b} f(x)dx \approx A_0 f(x_0) + A_1 f(x_1). \tag{1}$$

(b) What is the degree of (algebraic) precision of the formula (1) with the found coefficients and nodes?

(c) Prove that no Gassian quadrature formula of the form (1) can be exact for all polynomials of degree $\leq 4$.

Problem 3. Assume $b \in \mathbb{R}^m$, $A \in \mathbb{R}^{m \times n}$, $m < n$, and rank $A = m$. Then the system of linear equations $Ax = b$ has infinitely many solutions. Show how to obtain the minimum 2-norm solution of $Ax = b$ using a reduced SVD of $A$ given by $A = U\Sigma V^T$. 

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**Problem 4.** Assume that $A \in \mathbb{R}^{m \times m}$ is symmetric and positive definite. Let $A_1$ and $A_2$ be the matrices generated by two Cholesky steps:

$$
A = G_1 G_1^T, \quad A_1 = G_1^T G_1, \quad A_2 = G_2^T G_2,
$$

(2)

where $G_1, G_2 \in \mathbb{R}^{m \times m}$ are lower triangular with positive elements on the main diagonal. (The first equations in (2) and (3) are Cholesky factorizations of $A$ and $A_1$, respectively, while the second equations in (2) and (3) define $A_1$ and $A_2$.) Show that $A_1$ is symmetric and positive definite (this justifies the second Cholesky step). Also show that $A_2$ is symmetric and positive definite.

Let $B$ be the matrix produced by one QR step applied to $A$, that is,

$$
A = QR, \quad B = RQ,
$$

(4)

where $Q \in \mathbb{R}^{m \times m}$ is orthogonal and $R$ is upper triangular with positive elements on the main diagonal. Show that $B = A_2$.

**Problem 5.** Prove the Bauer-Fike Theorem:

Let $A$ be an $n \times n$ matrix with a complete set of linearly independent eigenvectors and suppose the $V^{-1} A V = D$, where $V$ is nonsingular and $D$ is diagonal. Let $\delta A$ be a perturbation of $A$ and let $\mu$ be an eigenvalue of $A + \delta A$. Then $A$ has an eigenvalue $\lambda$ such that

$$
|\mu - \lambda| \leq \kappa_p(V) \| \delta A \|_p, \quad 1 \leq p \leq \infty
$$

where $\kappa_p(V)$ is the $p$-norm condition number of $V$.

**Problem 6.** Consider the numerical method

$$
y_{n+1} = y_n + \frac{h}{2} [y'_n + y'_{n+1}] + \frac{h^2}{12} [y''_n - y''_{n+1}]
$$

for solving the initial value problem $y' = f(t, y), y(t_0) = y_0$, where $y'_n = f(t_n, y_n)$ and

$$
y''_n = \frac{\partial f(t_n, y_n)}{\partial t} + f(t_n, y_n) \frac{\partial f(t_n, y_n)}{\partial y}.
$$

Show that

i. the method is fourth-order, and

ii. the region of absolute stability contains the entire negative real axis.