All answers must be fully justified to receive full credit. All problems are of equal value.

1. Let $S_n$ denote the symmetric group on $n$ letters.
   
   (i) For the $k$-cycle $\sigma = (a_1, \ldots, a_k) \in S_n$ and $\tau \in S_n$, prove that $\tau \sigma \tau^{-1} = (\tau(a_1), \ldots, \tau(a_k))$.
   
   (ii) Use part (i) to find the number of distinct conjugacy classes in $S_5$.

2. Let $R$ be a unique factorization domain with quotient field $K$. Write $R[x]$ and $K[x]$ respectively, for the polynomial rings over $R$ and $K$. Fix $0 \neq f(x) \in R[x]$. Let $I$ denote the principal ideal $f(x)R[x]$ and $J$ denote the ideal $f(x)K[x] \cap R[x]$. Prove that there exists $0 \neq a \in R$ such that $I = aJ$, where $aJ := \{aj \mid j \in J\}$.

3. Let $F \subseteq K$ be a simple algebraic extension of fields, i.e., $K = F(\alpha)$, for some $\alpha \in K$ algebraic over $F$, and write $\overline{F}$ to denote the algebraic closure of $F$. Let $E(K/F)$ denote the number of distinct field homomorphisms $\sigma : K \to \overline{F}$ fixing $F$.
   
   (i) Prove that $E(K/F) \leq [K : F]$, where $[K : F]$ denotes the degree of $K$ over $F$.
   
   (ii) Give an example where $E(K/F) < [K : F]$.

4. Let $T : V \to W$ be a linear transformation of vector spaces over the field $F$, and assume $\dim(V) = \dim(W)$. Prove that there exist bases $B$ and $B'$ of $V$ and $W$ respectively, so that $[T]_{B'}^B$ is a diagonal matrix.

5. Let $V$ be a finite dimensional inner product space over $\mathbb{R}$ and $v_1, \ldots, v_n$ a basis for $V$. Let $c_1, \ldots, c_n \in \mathbb{R}$. Prove that there exists $v \in V$ such that $\langle v, v_i \rangle = c_i$, for all $1 \leq i \leq n$.

6. Consider the matrix $A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ -2 & 0 & -1 & -2 \\ 1 & -1 & 0 & 0 \end{bmatrix}$, with entries in $\mathbb{C}$. Find $J$, the Jordan canonical form for $A$ and an invertible matrix $P$ such that $J = P^{-1}AP$. 

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