Math Awareness Month Competition

University of Kansas, Department of Mathematics

2017 Examination for 6th-8th Grades

DIRECTIONS: You have 40 minutes for the five problems.
Show all of the necessary work and provide a complete justification for each answer.
Enclose each answer in a box. Solve each problem on a separate sheet of paper.
You are allowed to use a calculator but you are not allowed to borrow or interchange calculators during the test.

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1. If a small circle's diameter is a large circle's radius, what is the small circle's area in the percentage of the large circle's area?

   Solution: In the large circle, if $r = 2$, then the area would be 4π. Small circle then has $r = 1$ and the area is π. That's 25% of 4π.

2. If the sum of 2000 consecutive integers is 1000, what is the sum of the digits of the greatest of these 2000 integers?

   Solution: The 2000 integers $-999, 998, \ldots, 998, 999, 1000$ have a sum of 1000. The digit-sum of largest integer used is $1 + 0 + 0 + 0 = 1$.

3. The sum of the 3-digit numbers $35x$ and $4y7$ is divisible by 36. Find all possible ordered pairs (x, y).

   Solution: $(1, 7)$ and $(5, 3)$: $350 + x + 407 + 10y = 36j$ for some j. Since $757 = 36 * 21 + 1$, we should have $10y + x = 36k - 1$ for some k. Since $0 \le 10y + x \le 99$, we have $10y + x = 35$ or 71.

4. Let **mod** denote the modulo operation, the remainder after division, e.g.,

   $$5 \textbf{ mod } 2 = 1.$$

   What is the smallest number A such that for each number $N = 2, 3, 4, 5, 6, 7, 8$

   $$A \textbf{ mod } N = N - 1?$$

   Solution: $A = 839$.

   We need to calculate the *least common multiple* of these integers and subtract one, i.e., $\textbf{LCM}(2, 3, 4, 5, 6, 7, 8)$. Hence,

   $$A = \textbf{LCM}(2, 3, 4, 5, 6, 7, 8) - 1 = 840 - 1 = 839.$$

5. A stick was broken randomly into three pieces. What is the probability that a triangle can be built from the three parts?

Solution: The probability that a triangle can be build from given pieces is equal to $\frac{1}{4}$.

Assume that the length of the stick was 1. Let's denote the lengths of the left and the middle piece of the stick by a and b, respectively. Then $0 < a, b < 1$ and $a + b < 1$, and the third piece has length $1 - a - b$. The triangle can be constructed when the inequalities are: $b + (1 - a - b) > a$, $a + (1 - a - b) > b$ and $a + b > 1 - a - b$, i.e., $\frac{1}{2} > a$, $\frac{1}{2} > b$ and $a + b > \frac{1}{2}$. This corresponds to the shaded area in the figure below. The probability that a triangle can be build from given pieces is the ratio of the surface of the shaded area to the surface of the entire figure given conditions $0 < a, b < 1$ and $a + b < 1$ and it is equal to $\frac{1}{4}$.