Math Awareness Month Competition  
2013 Examination for 9 -12th Grades

**DIRECTIONS:** [40 Minutes - 5 Questions] Start each new problem on a separate page.  
**Show your work!** Answers must be exact. You are allowed to use a calculator.  
You are not allowed to borrow or interchange calculators during the test.

1. What is the smallest integer $N > 1$ such that for any positive integer $a$, $a^N$ and $a$ have the same last digit?

2. Find all positive integers $n$ such that $n^4 + n^2 + 1$ is a prime number.

3. Person A has 6 fair coins and Person B has 5 fair coins. Person A wins if he flips more heads than B does, otherwise B wins. What is the probability of A winning?

4. Let $A$ be a unit square. What is the largest area of a triangle whose vertices lie on the perimeter of $A$? Justify your answer.

5. Let $a, b, c$ be three distinct real numbers. Suppose that

$$\frac{a^2}{(b - c)^2} + \frac{b^2}{(c - a)^2} + \frac{c^2}{(a - b)^2} = 2.$$ 

Find $\frac{a}{b - c} + \frac{b}{c - a} + \frac{c}{a - b}$. 


Answers:

1. What is the smallest integer \( N > 1 \) such that for any positive integer \( a \), \( a^N \) and \( a \) have the same last digit?
   [Answer: Letting \( a = 2 \) shows that \( N > 4 \). Thus \( N = 5 \) by brute force checking (more clever methods are fine).]

2. Find all positive integers \( n \) such that \( n^4 + n^2 + 1 \) is a prime number.
   [Answer: \( n^4 + n^2 + 1 = n^4 + 2n^2 + 1 - n^2 = (n^2 + n + 1)(n^2 - n + 1) \). So one of the factors has to be 1. It follows that \( n = 1 \).]

3. Person A has 6 fair coins and Person B has 5 fair coins. Person A wins only if he flips more heads than B does, otherwise B wins. What is the probability of A winning?
   [Answer: \( 1/2 \). Imagine that A and B each toss 5 times. There is a certain probability \( p \) that A is ahead, and by symmetry the same probability \( p \) that B is ahead. The probability they are tied after 5 tosses is \( 1 - 2p \). Thus the probability that A wins is 
   \[ p + \frac{1}{2}(1 - 2p) = \frac{1}{2}. \]

4. Let \( A \) be a unit square. What is the largest area of a triangle whose vertices lie on the perimeter of \( A \)? Justify your answer.
   [Answer: \( 1/2 \). We claim that such largest area can be obtained with all vertices of the triangle also vertices of \( A \). Pick a vertex of the triangle that is not vertex of \( A \), call it \( V \). Fix the opposite triangle edge, and note that when we move \( V \) in one of the two directions along the edge of \( A \) it is on, the height would not decrease. So one can make \( V \) a vertex of \( A \). From here it is easy to check that the answer is \( 1/2 \).]

5. Let \( a, b, c \) be three distinct real numbers. Suppose that
   \[ \frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(a-b)^2} = 2. \]
   Find \( \frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b} \).
   [Answer: \( 0 \). Let \( x = \frac{a}{b-c}, \ y = \frac{b}{c-a}, \ z = \frac{c}{a-b} \). We observe that \( (x+1)(y+1)(z+1) = (x-1)(y-1)(z-1) \), so \( xy + yz + zx = -1 \). Thus \( (x+y+z)^2 = x^2 + y^2 + z^2 + 2(xy + yz + zx) = 0 \).]