Math Awareness Month Competition
2013 Examination for 9 -12th Grades

DIRECTIONS: [40 Minutes - 5 Questions] Start each new problem on a separate page.
Show your work! Answers must be exact. You are allowed to use a calculator.
You are not allowed to borrow or interchange calculators during the test.

1. What is the smallest integer $N > 1$ such that for any positive integer a, a^N and a have the same last digit?

2. Find all positive integers n such that $n^4 + n^2 + 1$ is a prime number.

3. Person A has 6 fair coins and Person B has 5 fair coins. Person A wins if he flips more heads than B does, otherwise B wins. What is the probability of A winning?

4. Let A be a unit square. What is the largest area of a triangle whose vertices lie on the perimeter of A? Justify your answer.

5. Let a, b, c be three distinct real numbers. Suppose that

$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(a-b)^2} = 2.$$

   Find $\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b}$.

Answers:

1. What is the smallest integer $N > 1$ such that for any positive integer a, a^N and a have the same last digit?
[Answer: Letting $a = 2$ shows that $N > 4$. Thus $N = 5$ by brute force checking (more clever methods are fine).]

2. Find all positive integers n such that n^4+n^2+1 is a prime number.
[Answer: $n^4+n^2+1 = n^4+2n^2+1-n^2 = (n^2+n+1)(n^2-n+1)$. So one of the factors has to be 1. It follows that $n = 1$.]

3. Person A has 6 fair coins and Person B has 5 fair coins. Person A wins only if he flips more heads than B does, otherwise B wins. What is the probability of A winning?
[Answer: 1/2. Imagine that A and B each toss 5 times. There is a certain probability p that A is ahead, and by symmetry the same probability p that B is ahead. The probability they are tied after 5 tosses is $1-2p$. Thus the probability that A wins is
$$p + \frac{1}{2}(1-2p) = \frac{1}{2}.]$$

4. Let A be a unit square. What is the largest area of a triangle whose vertices lie on the perimeter of A? Justify your answer.
[Answer: 1/2. We claim that such largest area can be obtained with all vertices of the triangle also vertices of A. Pick a vertex of the triangle that is not vertex of A, call it V. Fix the opposite triangle edge, and note that when we move V in one of the two directions along the edge of A it is on, the height would not decrease. So one can make V a vertex of A. From here it is easy to check that the answer is 1/2.]

5. Let a, b, c be three distinct real numbers. Suppose that
$$\frac{a^2}{(b-c)^2} + \frac{b^2}{(c-a)^2} + \frac{c^2}{(a-b)^2} = 2.$$
Find $\frac{a}{b-c} + \frac{b}{c-a} + \frac{c}{a-b}$.
[Answer: 0. Let $x = \frac{a}{b-c}$, $y = \frac{b}{c-a}$, $z = \frac{c}{a-b}$. We observe that $(x+1)(y+1)(z+1) = (x-1)(y-1)(z-1)$, so $xy+yz+zx = -1$. Thus $(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx) = 0$.]