

Math Awareness Month Competition 2010

Solutions for 7th-9th Grades

1. The average of the ages of the mother, the father, and their three children is 21, while the average of the children is 11. How old is the father if he is 4 years older than the mother?

[Solution: Let x = the age of the father. Then $x + (x - 4) + 5 \cdot 11 = 5 \cdot 21$. Thus, the age of the father is **38**.]

2. The sum of the digits of a given number a is the same as the sum of the digits of the number $2a$. Show that a is divisible by 9.

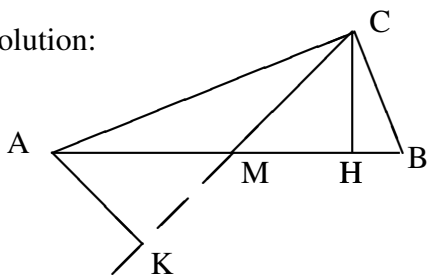
[Solution: Let x be a number and $S(s)$ denote the sum of the digits of x ; we know that $x - S(x)$ is divisible by 9. In particular, $a - S(a)$ and $2a - S(2a)$ are divisible by 9. As $S(a) = S(2a)$, $2a - a = (2a - S(2a)) - (a - S(a))$ is divisible by 9. It follows that a is divisible by 9.]

3. Show that if $|x| < 1$ and $|y| < 1$, then $|x - y| < 1 - xy$.

[Solution: Since $(x - y)^2 = x^2 - 2xy + y^2$, $(1 - xy)^2 = 1 - 2xy + x^2y^2$, we need to show that $x^2 - 2xy + y^2 < 1 - 2xy + x^2y^2$ or $x^2 - 2xy + y^2 - (1 - 2xy + x^2y^2) = x^2(1 - y^2) - (1 - y^2) = (x^2 - 1)(1 - y^2) < 0$. But $(x^2 - 1)(1 - y^2) < 0$ is true as $x^2 < 1$ and $y^2 < 1$.]

4. Consider a right triangle ABC , where the hypotenuse is AB and $AC > BC$. If the altitude from the vertex C in this triangle is $CH = 6$ cm and if M is the midpoint of AB , find the length in centimeters of the altitude from the vertex A in the triangle AMC .

[Solution:



$CM = \frac{1}{2}AB = AM$ (since $\angle ACM$ is a right angle and M is the midpoint of AB). It follows that $\triangle AMK \cong \triangle CMH$. A consequence of this congruence is that $AK = CH = 6$ centimeters.]

5. Show that the sum

$$S = \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots + \frac{1}{(2k+1)^2} + \cdots + \frac{1}{2011^2}$$

is less than $\frac{1}{4}$.

[Solution: Notice that $\frac{1}{(2k+1)^2} < \frac{1}{(2k+1)^2-1} = \frac{1}{2k(2k+2)} = \left(\frac{1}{4}\right)\left(\frac{1}{k} - \frac{1}{k+1}\right)$. So, $S <$
 $\left(\frac{1}{4}\right)\left(\frac{1}{1} - \frac{1}{2}\right) + \frac{1}{4}\left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \frac{1}{4}\left(\frac{1}{1005} - \frac{1}{1006}\right) = \frac{1}{4}\left(1 - \frac{1}{1006}\right) < \frac{1}{4}$.]