Qualifying Exam in Analysis
August 12, 1999

Work all six problems below.

1. Let \( \{x_1, x_2, \ldots \} \) be a convergent real sequence with limit \( x \). Let \( \{a_n\} \) be the corresponding sequence of averages, that is \( a_n = (x_1 + \ldots + x_n)/n \) for each \( n \). Show that \( \lim_n a_n = x \).

2. Give an example of a uniformly convergent sequence \( \{f_n\} \) of continuously differentiable functions on \([0,1]\) such that the sequence \( \{f'_n\} \) of derivatives fails to converge pointwise on \([0,1]\).

3. Let \( f \) be a continuously differentiable real function defined on an open ball in \( \mathbb{R}^k \) about the vector \( \vec{a} \) in \( \mathbb{R}^k \). Show that if \( \nabla f(\vec{a}) \) is nonzero, then for every vector \( \vec{v} \) in \( \mathbb{R}^k \) orthogonal to \( \nabla f(\vec{a}) \) there is a continuously differentiable \( \mathbb{R}^k \)-valued function \( \theta \) defined on an open interval about 0 in \( \mathbb{R} \) such that \( \theta(0) = \vec{a}, \ \theta'(0) = \vec{v} \), and the composition \( f \circ \theta \) is constant. (Hint: Let \( \vec{w} = \nabla f(\vec{a}) \) and apply the implicit function theorem to the real function \( g \) defined in a neighborhood of \((0,0)\) in \( \mathbb{R}^2 \) by \( g(s,t) = f(\vec{a} + s\vec{v} + t\vec{w}) \).)

4. Show that if the graph of a bounded real function on \( \mathbb{R} \) is closed in \( \mathbb{R}^2 \), then the function is continuous. (Remark: The assumption that the function be bounded is essential here.)

5. Let \( X \) be the set of all bounded real sequences \( \vec{x} = \{x_1, x_2, \ldots \} \) with metric \( d \) defined by

\[
d(\vec{x}, \vec{y}) = \sup_n |x_n - y_n|.
\]

The subset \( E \) consisting of all sequences \( \vec{x} \) such that \( d(\vec{x}, \vec{0}) \leq 1 \) is plainly a closed subset of \( X \). Is \( E \) also compact? (Of course you must say why or why not.)

6. (a) Give an example of a continuous real function on \( \mathbb{R} \) that is differentiable everywhere except at 0.

(b) Show that if \( f \) is a continuous real function on \( \mathbb{R} \) differentiable on \( \mathbb{R} \setminus \{0\} \) such that \( \lim_{x \to 0} f'(x) \) exists, then \( f \) is differentiable at 0 as well.