1. (20 pts) Let $E$ be the splitting field of $X^3 - 5$ over the rational numbers $\mathbb{Q}$. Find (with proof) the Galois group of $E$ over $\mathbb{Q}$.

2. (10 pts) Factor $2$ into irreducibles in the Guassian integers, $R = \mathbb{Z}[i]$. Justify your factorization.

3. (15 pts) Let $\alpha$ be a complex number and suppose that

$$\alpha^n + c_1 \alpha^{n-1} + \ldots + c_{n-1} \alpha + c_n = 0$$

where $c_1, ..., c_n \in \mathbb{Z}$, the integers. Prove that the minimal polynomial of $\alpha$ over the rational numbers has integer coefficients.

4. (15 pts) Let $R \subseteq S$ be Noetherian rings. Suppose that $S$ is finitely generated as an $R$-module, generated by $n$ elements. Let $m$ be a maximal ideal of $R$. Prove that there are at most $n$ maximal ideals of $S$ lying over $R$.

5. 
   a) (10 pts) State and prove the going up theorem.
   b) (10 pts) State and prove the Hilbert basis theorem.

6. (20 pts) Let $R$ be a Noetherian ring and suppose that $I \subseteq J$ are two ideals of $R$. Assume for all primes $P$ associated to $I$ that $J_P \subseteq I_P$. Prove that $I = J$. 