1. Show that the abelian group \((\mathbb{Q}, +)\) cannot be written as a direct sum of proper subgroups.

2. Let \(G\) be a group. (a) Show that \(G\) is abelian if and only if there exists an automorphism \(\tau : G \to G\) satisfying \(\tau(g) = g^{-1}\), for all \(g \in G\).

(b) Suppose that \(G\) is finite and that \(\tau\) is an automorphism of \(G\) satisfying the following properties: \(\tau^2 = 1\) and \(\tau\) fixes only the identity element of \(G\). Show that \(G\) is abelian. (Hint: Consider the set \(\{\tau(x)x^{-1} \mid x \in G\}\).)

3. Let \(R\) be a commutative ring and \(J \subseteq R\) an ideal. Show that there is a one-to-one correspondence between the ideals of \(R\) containing \(J\) and the ideals of \(R/J\). Use the correspondence to show that \(J\) is a maximal ideal if and only if \(R/J\) is a field.

4. Let \(f(X) := X^3 + X + 1 \in \mathbb{Q}[X]\). Show that \(f(X)\) is irreducible. Let \(\alpha\) be a root of \(f(X)\). Set \(a := 1 + \alpha + \alpha^2\) and \(b := 3 + 2\alpha + \alpha^2\). Express \(ab^{-1}\) as a \(\mathbb{Q}\)-linear combination of \(1, \alpha, \alpha^2\).

5. Let \(A\) be an \(n \times n\) matrix with entries in \(\mathbb{C}\). (a) Show \(A = D + N\), for some diagonalizable matrix \(D\) and nilpotent matrix \(N\) satisfying \(DN = ND\).

(b) Suppose \(A = D + N = D' + N'\), where \(D, D'\) are diagonal matrices and \(N, N'\) are nilpotent matrices satisfying \(DN = ND\) and \(D'N' = N'D'\). Show that \(NN' = N'N\) and deduce that \(N - N'\) is also nilpotent.

(c) For \(D, D', N, N'\) in part (b), show \(D = D'\) and \(N = N'\).

6. Let \(f(X)\) be a monic polynomial in \(\mathbb{F}[X]\) and denote by \(C_f\) the companion matrix of \(f(X)\). Show that \(f(X)\) equals the characteristic polynomial of \(C_f\). Discuss the role of companion matrices in the rational canonical form theorem.