Probability

1. An urn $U_1$ contains 2 white balls and 3 black balls. A second urn $U_2$ contains 2 white balls and 3 red balls. We draw a ball from $U_2$ and put it in $U_1$. Then, we draw two balls from $U_1$. Find the probability that $U_1$ has no red balls, assuming that the two balls drawn from $U_1$ are white and black.

2. Let $X$ and $Y$ two random variables with joint density function
   \[ f(x, y) = \begin{cases} \frac{x^2}{2} e^{-xy} & \text{if } -y < x < y, \quad y > 0 \\ 0 & \text{otherwise} \end{cases} \]
   a) Find the marginal probability density functions of $X$ and $Y$.
   b) Let $U = \frac{X + Y}{2}$ and $V = \frac{Y - X}{2}$. Are $U$ and $V$ independent? Justify your answer. Find the probability density functions of $U$ and $V$.
   c) Find the probability density function of $Y$ given $X$.

3. Let \( \{X_n, n \geq 1\} \) be a sequence of independent and identically distributed random variables with zero mean and \( E(X_n^2) = \sigma^2 < \infty \). Find the limit in distribution of the sequence
   \[ \sqrt{n} \frac{X_1X_2 + X_3X_4 + \cdots + X_{2n-1}X_{2n}}{X_1^2 + \cdots + X_{2n}^2} \]

4. Let \( \{X_n, n \geq 1\} \) be a sequence of independent random variables such that
   \[ P(X_n = \alpha) = p, \quad P(X_n = \frac{1}{\alpha}) = 1 - p, \]
   with \( 0 < \alpha < 1, \quad 0 < p < 1 \).
   a) Find the distribution of the random variable \( Y_n = \max(X_1, \ldots, X_n) \).
   b) Find the limit in probability of the sequence \( Y_n \).
   c) Let \( N \) be a random variable, independent of the sequence \( X_n \), such that
      \[ P(N = n) = (1 - \gamma)\gamma^{n-1}, \quad n \geq 1. \]
      Find the probability function of the random variable \( Y_N \).
Statistics

1. Suppose that $X_1, \ldots, X_n$ is a random sample from the density function

$$f(x, \theta) = \frac{1}{\theta} \left( \frac{1}{x} \right)^{1+\theta} 1_{[1, \infty)}(x),$$

where $\theta > 0$ is an unknown parameter.

a) Find a maximum likelihood estimator (MLE) for $\theta$.

b) Find the Cramer-Rao bound and the Fisher information function for the parameter $\theta$. Is the MLE efficient?

2. 1000 randomly selected people were asked about their opinion between two candidates A and B. 615 stated a preference for candidate A. Find a confidence interval with 95% for the proportion of votes that A will have. What should be the size of the sample to reduce the length of this interval to one half?

3. Consider a random sample of size $n$ of a normal distribution with mean 1 and variance $\sigma^2$.

a) Find an UMP (uniformly most powerful) test at level $\alpha = 0.05$ for the hypotheses

$$H_0 : \sigma^2 = \sigma^2_0$$
$$H_1 : \sigma^2 = \sigma^2_0/2,$$

where $\sigma^2_0$ is known.

b) Find the power of this test, assuming $n = 50$.

4. The number of typos in 100 pages of a book is:

<table>
<thead>
<tr>
<th>Typos</th>
<th>Frequency</th>
</tr>
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<tbody>
<tr>
<td>0</td>
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<td>25</td>
</tr>
<tr>
<td>2</td>
<td>8</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
</tr>
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Can we assume that the distribution of the typos is Poisson? Justify your answer.