Math 725, Spring 2006
Problem Set #1
Due date Wednesday, February 1 (in class)

Note: Subsequent problem sets will probably contain fewer, and more challenging, problems than this one. Right now, the goal is to become comfortable with all the definitions, and to get some practice in the kinds of arguments necessary in graph theory.

#1. [West 1.1.5, modified] Show that the statement “If $G$ is a 2-regular simple graph, then $G$ is a cycle” is false, by exhibiting an explicit counterexample. What additional condition on $G$ makes the statement true?

#2. Problem 1.1.9 in West.

#3. A subgraph $H \subseteq G$ is called a spanning subgraph if $V(H) = V(G)$.
   (a) Let $G$ be an arbitrary graph. How many spanning subgraphs does $G$ have?
   (b) Prove that $G$ is connected if and only if it has a connected spanning subgraph.
   (c) Use (b) to solve problem 1.1.10 in West. (Hint: If $G$ is disconnected, what spanning subgraph must its complement have?)

#4. [West 1.1.30, shortened] Let $G$ be a graph on vertex set $[n]$, and let $A = A(G)$ be the ($n$-by-$n$) adjacency matrix of $G$. What does the entry of $A^2$ in position $(i,j)$ tell you about $G$?

#5. Problem 1.1.25 in West.

#6. Let $G$ be a graph and $H$ a subgraph of $G$. Prove that if $H$ is not bipartite, then neither is $G$. Deduce the “only if” direction of Theorem 1.2.18.

#7. Recall that the $n$-dimensional cube $Q_n$ is the graph whose vertices are the bit strings of length $n$, with two vertices adjacent if and only if they differ in exactly one bit. Show that $Q_n$ is bipartite, by exhibiting an explicit bipartition.

#8. Problem 1.2.5 in West.

#9. Problem 1.2.7 in West.

#10. Problem 1.2.22 in West.

Bonus problem: Let $R_n$ be the graph on the bit strings of length $n$, in which two bit strings are adjacent if and only if they agree in exactly one bit. Show that $R_n \cong Q_n$ if and only if $n$ is even.