#1. Let $G$ be a complete graph with vertices $\{v_1, \ldots, v_n\}$. Define a weight function on $E(G)$ by
$$\text{wt}(v_i v_j) = \max\{i, j\}.$$ 
Prove that $G$ has exactly $(n - 1)!$ distinct minimum-weight spanning trees.

#2. Let $k \geq 1$, and let $G$ be a $k$-regular bipartite graph. Prove that $G$ has a perfect matching.

#3. Let $G$ be the following graph with vertices $\{v_1, \ldots, v_7\}$.

![Graph Diagram](image)

(3a) Construct a source-sink network $N$ such that a maximum flow in $N$ corresponds to a maximum matching in $G$.

(3b) Construct a source-sink network $N'$ such that a maximum flow in $N'$ corresponds to a maximum family of pairwise internally disjoint $v_2, v_4$-paths in $G$.

#4. Let $G$ be a planar graph, and let $G^*$ be its planar dual.

(4a) What invariant of $G^*$ equals the girth of $G$? Explain your answer.

(4b) Prove that if $G$ has a cut-vertex, then so does $G^*$.
#5. The octahedron is the graph $O$ shown in the following figure.

(5a) Prove that $O$ has chromatic number 3. Your answer should not make use of the chromatic polynomial.

(5b) Now calculate the chromatic polynomial of the octahedron. (Hint: $O$ can be formed by deleting a perfect matching from $K_6$.) Use your answer to give another proof that $O$ has chromatic number 3.

#6. Let $G$ be a connected graph. Recall that the Tutte polynomial of $G$ is defined by

$$T_G(x, y) = \sum_{A \subseteq E(G)} (x - 1)^{r(E) - r(A)} (y - 1)^{|A| - r(A)}$$

where $r(A)$ denotes the size of a maximum acyclic subset of $A$.

(6a) What does the number $T_G(2, 2)$ tell you about $G$?

(6b) Explain how to use the Tutte polynomial to calculate the number of acyclic subsets of $E(G)$.

(6c) (Extra credit) Explain how to use the Tutte polynomial to find the edge-connectivity $\kappa'(G)$. 