

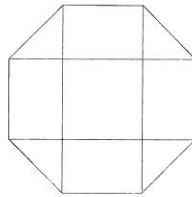
**Math Awareness Month Competition**  
**University of Kansas, Department of Mathematics**  
**2017 Examination for 9th-12th Grades**

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**DIRECTIONS:** You have 40 minutes for the five problems.  
Show all of the necessary work and provide a complete justification for each answer.  
Enclose each answer in a box. Solve each problem on a separate sheet of paper.  
You are allowed to use a calculator but you are not allowed to borrow or interchange calculators during the test.

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1. A dart board is a regular octagon divided into regions as shown. Suppose that a dart thrown at the board is equally likely to land anywhere on the board. What is the probability that the dart lands outside the center square?



**Solution:** Assume the octagon's edge is 1. Then the corner triangles have area  $1/4$  each. The four rectangles have area  $\sqrt{2}/2$  each. The center square has area 1. Then the total area is  $2 + 2\sqrt{2}$ , and the probability is  $(1 + 2\sqrt{2}) / (2 + 2\sqrt{2}) = (3 - \sqrt{2})/2$ .

2. Let  $f(x) = 10^{10x}$ ,  $g(x) = \log_{10}(x/10)$ ,  $h_1(x) = g(f(x))$ , and  $h_n(x) = h_1(h_{n-1}(x))$  for integers  $n \geq 2$ . What is the sum of the digits of  $h_{2017}(1)$ ?

**Solution:** Calculation yields that  $h_1(x) = 10x - 1$  and  $h_n(x) = 10^n x - \sum_{k=0}^{n-1} 10^k$ . Hence  $h_n(1)$  is an  $n$ -digit integer whose unit digit is 9 and other digits are all 8's. Then the sum is  $8 \times 2016 + 9 = 16,137$ .

3. Amy, Beth and Jo listen to four different songs and discuss which ones they like. No song is liked by all three. Furthermore, for each of the three pairs of the girls, there is at least one song liked by those two girls but disliked by the third. In how many different ways is this possible?

**Solution:** Case 1: Each song is liked by two of the girls. Then one of the three pairs of girls likes one of the six possible pairs of songs, one of the remaining

pairs of girls likes one of the remaining two songs, and the last pair of girls likes the last song. This is  $3 \times 6 \times 2 = 36$  ways.

Case 2: Three songs are each liked by a different pair of girls, and the fourth song is liked by at most one girl. The songs can be assigned to these four cases in  $4!$  ways, and the last song can be liked by A, B, J or nobody. This is  $4! \times 4 = 96$  ways.

The total is  $36 + 96 = 132$  ways.

4. Find all parameters  $t$ , where  $0 \leq t \leq \pi$ , for which the equation  $\sin(x + t) = 1 - \sin(x)$  does not have a solution.

**Solution:** The equation is equivalent to  $\sin(x + t) + \sin(x) = 1$  and further to  $2 \sin(x + t/2) \cos(t/2) = 1$ . No solution if  $\cos(t/2) = 0$ . Otherwise, we get  $\sin(x + t/2) = 1/(2 \cos(t/2))$ , which has a solution iff  $-1 \leq 1/(2 \cos(t/2)) \leq 1$ , that is  $-2 \leq 1/(\cos(t/2)) \leq 2$ . Here, the left inequality holds because  $0 \leq t \leq \pi$  implies  $\cos(t/2) \geq 0$ . The right inequality holds iff  $\cos(t/2) \geq 1/2$ . Therefore, no solution if  $0 \leq \cos(t/2) < 1/2$ , that is  $\frac{\pi}{3} < t/2 \leq \frac{\pi}{2}$ . The answer is  $\frac{2\pi}{3} < t \leq \pi$ .

5. How many different boxes exist with the following properties: At least one pair of opposite sides are squares, the length of edges are integers measuring in inches, and the total surface in square inches equals the volume in cubic inches.

**Solution:** Denote  $n$  the length of the sides of the square, and  $k$  the height. The equal surface and volume gives  $n^2k = 2n^2 + 4nk$ , which can be written as  $(n - 4)(k - 2) = 8$ . Since  $n - 4 > -4$  and  $k - 2 > -2$ , we have only  $(12, 3); (8, 4); (6, 6); (5, 10)$ , that is four boxes.