Math Awareness Month Competition 2011
Solutions for 10–12th Grades

1. Two numbers x, y are chosen randomly (without replacements) from the set of integers $S = \{1, 2, 3, \cdots, 16\}$. What is the probability that the numbers $(x, y, 5)$ form the sides of a right triangle?

[Solution: There are 4 possibilities $(3, 4), (4, 3), (12, 13), (13, 12)$, so the probability is $\frac{4}{15\times 16} = 1/60$.]

2. A parallelogram has perimeter 40 centimeters and area equals to 100 square centimeters. What are the sides of this parallelogram?

[Solution: Let x, y be the sides and A be the area. It is easy to see that $xy \geq A$, with equality if and only if the sides are perpendicular (in other words, we have a rectangle). By assumption $x + y = 20$. Since $(x - y)^2 = x^2 + y^2 - 2xy$ is always non-negative we have:

$$20^2 = (x + y)^2 \geq 4xy \geq 4A = 4 \times 100 = 400$$

Since equality happens, we must have $x = y = 10$.]

3. Let a, b be the real roots of the equation $x^2 - 3^{2011}x + 3^{4020}$. Find $\log_3 \frac{a^3+b^3}{2}$

[Solution: We have $a^3 + b^3 = (a + b)^3 - 3ab(a + b) = 3^{6033} - 3^{6032} = 2 \times 3^{6032}$. The answer is 6032.]

4. Find the last two digits of 2^{1000}.

[Solution: Note that $2^{10} = 1024$, so we have to find the last two digits of 24^{100}. Since 24^2 ends in 76, and 76×24 ends in 24, the last two digits of powers of 24 will be a periodic sequence $24, 76, 24, 76, 24, \cdots$. The answer is 76.]

5. Solve the equation $\cos(x) \cos(2x) \cos(4x) = 1/8$.

[Solution: Multiply both sides by $\sin(x)$ and use the double formula $2 \sin(x) \cos(x) = \sin(2x)$ repeatedly we get $\sin(8x) = \sin(x)$. So $8x = x + 2k\pi$ or $8x = (2k + 1)\pi - x$. Therefore $x = 2k\pi/7$ or $x = (2k + 1)\pi/9$ for k any integer.]