Math Awareness Month Competition 2010

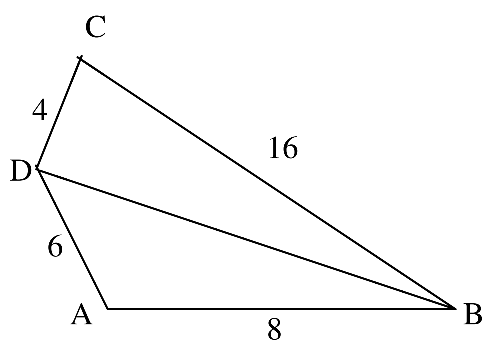
Solutions for 10th-12th Grades

1. Solve the equation: $\log_2^2 x + (x-1)\log_2 x = 6 - 2x.$

[Solution: Let $y = \log_2 x$; so, $x = 2^y$. The equation becomes $y^2 - (2^y - 1)y = 6 - 2 \cdot 2^y$. That is, we have: 0 = $y^2 - (2^y - 1)y - 6 + 2 \cdot 2^y = (y+2)(y + 2^y - 3)$. So, $y = -2$ and $x = \frac{1}{4}$ or $y + 2^y - 3 = 0$ and $y = 1, x = 2$. The solutions are $x = \frac{1}{4}$ and $x = 2$.]

2. The sides of the polygon $ABCD$ are given in centimeters as follows: $AB = 8$ cm, $BC = 16$ cm, $CD = 4$ cm, and AD cm. What is the length of the diagonal BD if it is known that it is an integer number in centimeters.

[Solution:



From $\triangle ABD$, $BD < 6+8 = 14$. From $\triangle BCD$, $BD > 16-4 = 12$. So, $BD = 14$ centimeters.]

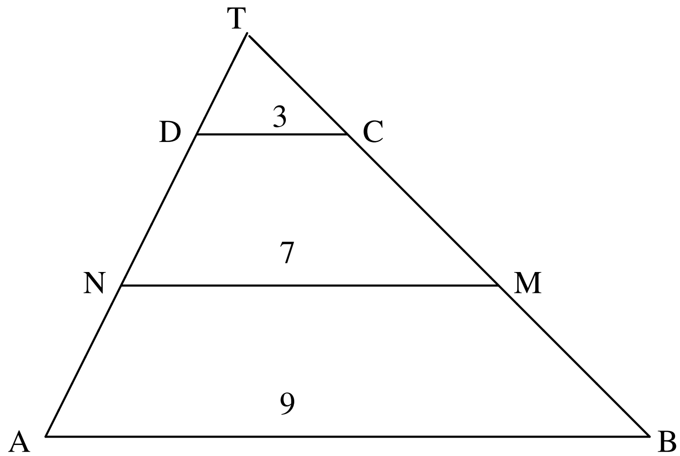
3. Solve the equation: $(2\sin x - 1)(\sin x + 1) = \cos x.$

[Solution: From $\cos x = 2\sin^2 x + 2\sin x - \sin x - 1 = 2\sin^2 + \sin x - (\sin^2 x + \cos^2 x)$, we get $\sin^2 x - \cos^2 x = \cos x - \sin x$ and $(\sin x + \cos x)(\sin x - \cos x) = -(\sin x + \cos x)$. So, $\sin x - \cos x = 0$ or $\sin x + \cos x = 0$. The solutions are $x = \frac{\pi}{4} + k\pi$ and $x = \frac{-\pi}{2} + 2k\pi$ where k is an integer.]

4. The bases for a trapezoid $ABCD$ are $AB = 9$ and $CD = 3$ and the other two sides are AD and BC. M is a point on the side BC and N is a point on the side of AD such that MN is parallel to AB and CD and $MN = 7$. Find the quotient of the area of $ABMN$ and the area of $NMCD$, i.e., compute

$$S_{ABMN} : S_{NMCD} =?$$

[Solution:



Let T be the point of intersection of AD and BC. Now, $\dfrac{S_{\triangle ABT}}{S_{\triangle NMT}} = \dfrac{81}{49}$ and $\dfrac{S_{\triangle NMT}}{S_{\triangle DCT}} = \dfrac{49}{9}$.

$$\frac{S_{ABMN}}{S_{NMCD}} = \frac{S_{\triangle ABT} - S_{\triangle NMT}}{S_{\triangle NMT} - S_{\triangle DCT}} = \frac{\frac{81}{49}S_{\triangle NMT} - S_{\triangle NMT}}{S_{\triangle NMT} - \frac{9}{49}S_{\triangle NMT}} = \frac{\frac{32}{49}}{\frac{40}{49}} = \frac{4}{5}.$$]

5. If the equation

$$x^4 - ax^3 + 2x^2 - bx + 1 = 0$$

has a real root, show that

$$a^2 + b^2 \geq 8.$$

[Solution: Let y be a real root of the equation. Then $0 = y^4 - ay^3 + 2y^2 - by + 1$, and $0 = y^2 - ay + 2 - b\frac{1}{y} + \frac{1}{y^2}$. Completing the square, we get $(y - \frac{a}{2})^2 + (\frac{1}{y} - \frac{b}{2})^2 = \dfrac{a^2 + b^2 - 8}{4} \geq 0$. Thus, $a^2 + b^2 \geq 8$.]